

Lower Semicontinuity Concepts, Continuous Selections, and Set Valued Metric Projections

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A number of semicontinuity concepts and the relations between them are discussed. Characterizations are given for when the (set-valued) metric projection P_M onto a proximal subspace M of a normed linear space X is approximate lower semicontinuous or 2-lower semicontinuous. A geometric characterization is given of those normed linear spaces X such that the metric projection onto every one-dimensional subspace has a continuous $C_0(T)$ and $L_1(\mu)$ that have this property are determined. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Let M be a (linear) subspace of the (real) normed linear space X . For any $x \in X$, the (possibly empty) set of best approximations to x from M is defined by

$$P_M(x) := \{y \in M \mid \|x - y\| = d(x, M)\},$$

where $d(x, M) := \inf\{\|x - y\| \mid y \in M\}$. The set valued mapping $P_M: X \rightarrow 2^M$ thus defined is the **metric projection** onto M . The subspace M is said to be **proximal** (resp., **Chebyshev**) if $P_M(x)$ is nonempty (resp., is a singleton) for each $x \in X$. For example, every finite-dimensional subspace of a normed linear space or any closed subspace of a reflexive Banach space is proximal. Also, any proximal subspace of a strictly convex space is Chebyshev.

If M is finite-dimensional, then the metric projection $P_M: X \rightarrow 2^M$ is non-empty compact convex set valued and upper semicontinuous (see, e.g., [25]). Thus if it is Chebyshev, the singleton valued metric projection is continuous.

The most important semicontinuity concept for set valued mappings is that of **lower semicontinuity** (see Section 2), and the most important result is Michael's selection theorem: *if X is a paracompact Hausdorff topological space, Y is a Banach space, and $F: X \rightarrow 2^Y$ is a non-empty closed convex set valued and lower semicontinuous mapping, then F has a continuous selection; that is, there exists a continuous $s: X \rightarrow Y$ such that $s(x) \in F(x)$ for each $x \in X$ [21, 22].* In particular, if M is a finite-dimensional subspace of a normed linear space X and P_M is lower semicontinuous, then the metric projection $P_M: X \rightarrow 2^M$ has a continuous selection. There has been considerable interest in the existence of continuous selections for metric projections of which the references to this paper represent a small part. However, lower semicontinuity is neither a property which one can expect of a metric projection, nor is it a necessary condition for the existence of a continuous selection. Two of the authors were led to seek a weaker continuity condition that would be, for a metric projection, equivalent to the existence of a continuous selection; in [10] Deutsch and Kenderov formulated a concept, here called **approximate lower semicontinuity** (and in [10] called "almost" lower semicontinuity), and a related weaker concept of **k -lower semicontinuity** ($k = 2, 3, \dots$) which are necessary, but not in general sufficient conditions for the existence of a continuous selection. For a set valued mapping $F: X \rightarrow 2^Y$ between topological spaces X and Y , Brown [8] defined the **derived mapping** F' of F and a transfinite sequence of higher derived mappings. The derived mapping appears, without the name and in a particular context, in [7]; see also [2]. Przesławski and Rybiński [23], in

a narrower context, defined **weak lower semicontinuity** and some related notions. In fact, this same notion (under the name “quasi-lower” semicontinuity) was studied a few years earlier by Gutev [14]. The definitions of, and the relations between, these concepts are presented systematically in Section 2.

There are normed linear spaces in which metric projections onto finite-dimensional subspaces are more restricted. For example, there are spaces in which all metric projections onto finite-dimensional subspaces are lower semicontinuous (the spaces with property (P) of [6]). The spaces $C(T)$ of real continuous functions on a compact Hausdorff space T , and $L_1(\mu)$ of integrable functions on a measure space, have been approximation theoretically much studied. They have the property that a metric projection P_M onto a finite-dimensional subspace possesses a continuous selection if and only if it is approximate lower semicontinuous, and, in the $C(T)$ case, if and only if it is 2-lower semicontinuous. Zhivkov [27] and Brown [8] showed, independently and by rather different approaches, that this does not hold for all normed linear spaces X (contrary to the mistaken claim by Zhiqiang [26]). These results suggest that the concepts of approximate lower semicontinuity and 2-lower semicontinuity deserve further study. They are the subjects of Sections 3 and 4, respectively. Section 3 gives a characterization of approximate lower semicontinuity. Section 4 gives two characterizations of 2-lower semicontinuity of a metric projection onto a finite-dimensional subspace, one in terms of the geometry of the unit ball of the space, the other involving duality. Section 5 characterizes those normed linear spaces with the property that all metric projections onto one-dimensional subspaces have continuous selections, and shows that they are those spaces with the property that all metric projections onto finite-dimensional subspaces are approximate lower semicontinuous. A space $C(T)$ or $L_1(\mu)$ is such a space if and only if it is finite-dimensional. Section 5 concludes with a simple example of such a space in which there are one-dimensional subspaces with metric projections which are not lower semicontinuous.

2. LOWER SEMICONTINUITY CONCEPTS

Let X and Y be topological spaces and $F: X \rightarrow 2^Y$ a set valued mapping. The **essential domain** of F is defined to be

$$D(F) := \{x \mid F(x) \neq \emptyset\}.$$

Let us say that F is lower semicontinuous at $(x_0, y_0) \in X \times Y$ if $y_0 \in F(x_0)$ and $x_0 \in \text{int}\{x \mid F(x) \cap V \neq \emptyset\}$ for every neighborhood V of y_0 . F is called

lower semicontinuous (lsc) if $D(F) = X$ and F is lower semicontinuous at each point of the “graph”

$$\mathcal{G}(F) := \bigcup_{x \in X} \{ \{x\} \times F(x) \}$$

of F .

Brown [8] defined the **derived mapping** F' of a set valued mapping F :

$$F'(x) := \{ y \in F(x) \mid F \text{ is lsc at } (x, y) \}.$$

Thus F is lower semicontinuous if and only if $D(F) = X$ and $F' = F$. Higher order derived mappings are defined by transfinite induction:

$$F^{(0)} = F,$$

$$\text{if } \alpha \text{ is an ordinal, then } F^{(\alpha+1)} = (F^{(\alpha)})',$$

$$\text{if } \alpha \text{ is a limit ordinal, then } F^{(\alpha)} = \bigcap_{\beta < \alpha} F^{(\beta)},$$

where the intersection has its natural pointwise meaning.

Then $(\mathcal{G}(F^{(\alpha)} \mid \alpha \text{ an ordinal})$ is a decreasing transfinite sequence of subsets of $X \times Y$ and so is eventually constant. Consequently $(F^{(\alpha)} \mid \alpha \text{ an ordinal})$ is eventually constant; let F^* , the **stable derived mapping** of F , be its eventual value, and let $\text{order}(F)$ be $\min \{ \alpha \mid F^{(\alpha)} = F^* \}$. It is shown in [4] that every ordinal is the order of some set valued mapping.

A mapping $s: X \rightarrow Y$ is a **selection** for F if $s(x) \in F(x)$ for every $x \in X$ (one can write $s \in F$). If $s: X \rightarrow Y$ is a continuous selection then, obviously, $s \in F'$ and therefore $s \in F^*$.

Let Y now be restricted to be a metric space.

For any set or point A in Y denote by $B(A, \varepsilon)$ and $\bar{B}(A, \varepsilon)$ the open and closed ε -neighborhoods (if $\varepsilon > 0$) of A in Y ; that is,

$$B(A, \varepsilon) := \{ y \in Y \mid d(y, A) < \varepsilon \},$$

$$\bar{B}(A, \varepsilon) := \{ y \in Y \mid d(y, A) \leq \varepsilon \}.$$

If $x \in Y$, let

$$S(x, \varepsilon) := \{ y \in Y \mid d(y, x) = \varepsilon \}.$$

A subscript will be added to B or S , e.g. B_Y or S_Y , if there is need to specify the space Y .

F is said to be **approximate lower semicontinuous** (alsc) at x_0 if, for each $\varepsilon > 0$, there exists a neighborhood U of x_0 such that

$$\bigcap \{ B(F(x), \varepsilon) \mid x \in U \} \neq \emptyset.$$

Here we vary the terminology of [10], where approximate lsc was called "almost" lsc. It was proved there that F is also if and only if, for each $\varepsilon > 0$, there exists a continuous function $f = f_\varepsilon: X \rightarrow Y$ such that $f(x) \in B(F(x), \varepsilon)$ for all $x \in X$. The notions of approximate lower semicontinuity and derived mapping are related by the following implications:

$$F'(x) \neq \emptyset \Rightarrow F \text{ is also at } x.$$

$$F(x) \text{ is compact and } F \text{ is also at } x \Rightarrow F'(x) \neq \emptyset.$$

(The condition that $F(x)$ is compact cannot be omitted.)

One of the main results of [8] is that if Y is a normed linear space of finite dimension n and $F: X \rightarrow 2^Y$ is convex valued, then $F^{(n)} | \text{int } D(F^{(n)})$ is lower semicontinuous, and, therefore, if $D(F^{(n)}) = X$, then $F^* = F^{(n)}$.

The definition of also led naturally in [10] to that of k -lower semicontinuity. The set valued mapping F is **k -lower semicontinuous** (k -lsc) at x_0 if for each $\varepsilon > 0$, there exists a neighborhood U of x_0 such that

$$\bigcap_{i=1}^k B(F(x_i), \varepsilon) \neq \emptyset$$

for each choice of k points x_1, \dots, x_k in U . The definitions are related by the following consequence [10] of Helly's theorem. If Y is a normed linear space of finite dimension n and F is closed convex valued, then

$$F \text{ is also} \Leftrightarrow F \text{ is } (n+1)\text{-lsc.}$$

The last definition we consider here is due to Przesławski and Rybiński [23]. If Y is a metric space, then $F: X \rightarrow Y$ is **weakly lower semicontinuous** (wlsc) at $x_0 \in X$ if for each $\varepsilon > 0$ and each neighborhood U of x_0 there exists $x' \in U$ such that for each $z \in F(x')$

$$x_0 \in \text{int}\{x \mid F(x) \cap B(z, \varepsilon) \neq \emptyset\}.$$

Gutev [14] had considered the same notion earlier but under the name "quasi-lower" semicontinuity. The papers [23, 24] contain a number of related concepts, defined in less general circumstances, which are not considered here. It is clear that $\text{lsc} \Rightarrow \text{wlsc} \Rightarrow \text{also}$. Weak lower semicontinuity is related to the derived mapping by the following theorem. It was first established by Gutev [15, Theorem 2.1] by a different method.

THEOREM 2.1. *If X is a topological space, Y is a complete metric space, and $F: X \rightarrow 2^Y$ is closed valued and wlsc, then F' is lsc.*

Proof. It must be shown that $D(F') = X$ and that $F'' = F'$. A single argument applied to two cases achieves both.

Case 1. Let x_0 be any point of X and let $U_0 = X$.

Case 2. Let \bar{x}_0 be any point of X and $y_0 \in F'(\bar{x}_0)$. Let $\varepsilon > 0$ and let U_0 be an open neighborhood of \bar{x}_0 such that $F(x) \cap B(y_0, \varepsilon) \neq \emptyset$ for $x \in U_0$. Let x_0 be any point of U_0 .

Let $\eta > 0$ and $\eta_n = \eta/2^n$. We now choose inductively a decreasing sequence $(U_n)_{n \geq 0}$ of neighborhoods of x_0 , points $x'_n \in U_{n-1}$ and $z_n \in F(x'_n)$ for $n \geq 1$, according to the following specifications.

(i)_n If $n \geq 1$ and the neighborhood U_{n-1} of x_0 has been chosen, then by the wsc of F we can choose $x'_n \in U_{n-1}$ such that

$$x_0 \in \text{int}\{x \mid F(x) \cap B(z, \eta_n) \neq \emptyset\}$$

for each $z \in F(x'_n)$.

(ii)₁ In Case 1 choose $z_1 \in F(x'_1)$; in Case 2 choose $z_1 \in F(x'_1) \cap B(y_0, \varepsilon)$, which is possible by the choice of U_0 .

(ii)_n For $n \geq 2$, choose $z_n \in F(x'_n) \cap B(z_{n-1}, \eta_{n-1})$.

(iii)_n For $n \geq 1$ let

$$U_n = U_{n-1} \cap \{x \mid F(x) \cap B(z_n, \eta_n) \neq \emptyset\}.$$

By (i)_n the set U_n defined by (iii)_n is a neighborhood of x_0 .

By (iii)_{n-1} for $n \geq 2$, if $x \in U_{n-1}$, then $F(x) \cap B(z_{n-1}, \eta_{n-1}) \neq \emptyset$, so that $F(x'_n) \cap B(z_{n-1}, \eta_{n-1}) \neq \emptyset$ and the choice (ii)_n is possible.

Now by (ii)_n, $d(z_{n-1}, z_n) < \eta_{n-1}$. Thus

$$d(z_m, z_{m+k}) < \eta_m + \dots + \eta_{m+k-1} < 2\eta_m = 2^{-m+1}\eta$$

and (z_n) is a Cauchy sequence. Let $z^* = \lim z_n$. Then $d(z_n, z^*) \leq 2\eta_n$ and $z_n \in B(z^*, 2\eta_n)$. If $x \in U_n$ then, by (iii)_n, $F(x) \cap B(z^*, 3\eta_n) \neq \emptyset$, i.e. $d(z^*, F(x)) < 3\eta_n$. Now $x_0 \in U_n$ for all n , and $F(x_0)$ is closed, so $z^* \in F(x_0)$. It follows that $z^* \in F'(x_0)$. Applied to Case 1, this shows that $F'(x_0) \neq \emptyset$ for each $x_0 \in X$.

Next consider Case 2. By (ii)₁,

$$d(y_0, z^*) \leq d(y_0, z_1) + d(z_1, z^*) < \varepsilon + \eta,$$

so $d(y_0, F'(x_0)) < \varepsilon + \eta$. This holds for any $x_0 \in U_0$, and it follows that $y_0 \in F''(x_0)$. Thus $F'' = F'$ and the proof is complete. ■

Now let X be a paracompact Hausdorff space, let Y be a normed linear space of finite dimension n , and let $F: X \rightarrow 2^Y$ be compact convex valued. (All these conditions are fulfilled if X is a normed linear space, $Y = M$ is a subspace of X of dimension n and $F = P_M$ is the metric projection of X onto M .) Then all the implications described above hold good, and the basic conditions of Michael's selection theorem are satisfied by X and Y . The derived mappings of F are also all compact convex valued. In these circumstances F possesses a continuous selection if and only if $D(F^*) = X$ [8]. We now consider the following conditions (if $n > 1$, then there are $2n + 1$ of them).

- F is k -lsc ($k = 2, \dots, n$);
- F is also $\Leftrightarrow F$ is $(n + 1)$ -lsc $\Leftrightarrow D(F') = X$;
- $D(F^{(k)}) = X$ ($k = 2, \dots, n - 1$);
- F has a continuous selection $\Leftrightarrow D(F^*) = X \Leftrightarrow D(F^{(n)}) = X$;
- F is wlsc $\Rightarrow F'$ is lsc;
- F is lsc.

Each condition is implied by its successor. $D(F^{(n-1)}) = X \not\Rightarrow D(F^{(n)}) = X$ [8]. We expect that in general, when $n > 1$, all these $2n + 1$ conditions are distinct. In the case $n = 1$, the above implications yield the result [10]: *if $\dim Y = 1$, then F has a continuous selection if and only if F is als.*

There are further particular circumstances in which the latter equivalence holds. Let X be a normed linear space and $Y = M$ a finite-dimensional subspace of X . Then the implication

$$P_M \text{ is als} \Leftrightarrow P_M \text{ has a continuous selection}$$

holds in the following cases.

(1) $X = C(T)$, the space of real continuous functions on a compact Hausdorff space T , equipped with the uniform norm. This result was obtained independently by Li [18] and Fischer [12]. (This result also follows from later work of Li [19] where it also extends to $X = C_0(T)$, the space of real continuous functions which vanish at infinity on a locally compact Hausdorff space T .)

(2) $X = L^1(T, \mu)$, the space of either real [20] or complex [3] integrable functions on a measure space (T, μ) , equipped with the usual norm.

Even more dramatically, Fischer [13] showed that in $C(T)$,

$$P_M \text{ is 2-lsc} \Rightarrow P_M \text{ has a continuous selection.}$$

These results show that the spaces $C_0(T)$ and $L^1(T, \mu)$ are in a sense “nice” from the point of view of an interest in metric projections, and they suggest an investigation of such “niceness.” In Section 5 spaces X in which all metric projections onto one-dimensional subspaces possess continuous selections are considered.

3. APPROXIMATE LOWER SEMICONTINUITY

Throughout this section, X will denote a topological space and M a normed linear space of finite dimension n . We will consider mappings $P: X \rightarrow 2^M$ which are non-empty compact convex set valued and upper semicontinuous. If X is a normed linear space, M is a subspace of X , and $P = P_M$ is the metric projection onto M , then P satisfies the conditions.

The following notation and terminology will be used. The collection of all non-empty subsets of the normed linear space M which are compact and convex will be denoted $\mathcal{H}(M)$. The space $\mathcal{H}(M)$ will be endowed with the Hausdorff metric h ; that is,

$$h(A, B) := \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

for all A, B in $\mathcal{H}(M)$.

LEMMA 3.1. *Let $x_0 \in X$. Then the following statements are equivalent.*

1. P is not *alc* at x_0 ;
2. $P'(x_0) = \emptyset$;
3. For each $y_0 \in P(x_0)$, there exists a net $x_\alpha \rightarrow x_0$ and a $\delta > 0$ such that

$$d(y_0, P(x_\alpha)) \geq \delta \quad \text{for all } \alpha; \tag{3.1}$$

4. For each $y_0 \in P(x_0)$, there exists a compact convex set $A(y_0) \subseteq P(x_0) \setminus \{y_0\}$ and a net $x_\beta \rightarrow x_0$ such that

$$h(P(x_\beta), A(y_0)) \rightarrow 0. \tag{3.2}$$

Proof. The equivalence of (1) and (2) has been noted in Section 2. The equivalence of (2) and (3) follows immediately from the definition of the derived mapping.

To prove the implication (3) \Rightarrow (4), assume (3) holds and let $y_0 \in P(x_0)$. Choose a net $x_\alpha \rightarrow x_0$ and $\delta > 0$ such that (3.1) holds. Since $(P(x_\alpha))$ is a net of compact convex sets in M which is eventually bounded, the Blaschke selection theorem (e.g., see [16]) implies that there exists a subnet, which

we will denote $(P(x_\beta))$, and a compact convex set $A = A(y_0) \subseteq M$ such that $h(P(x_\beta), A) \rightarrow 0$. By the upper semicontinuity of P , for each $\varepsilon > 0$, $P(x_\beta) \subset B(P(x_0), \varepsilon)$ eventually. Thus

$$A \subseteq B(P(x_\beta), \varepsilon) \subseteq B(P(x_0), 2\varepsilon)$$

eventually. Since ε was arbitrary and $P(x_0)$ is closed, $A \subseteq P(x_0)$.

If $y_0 \in A$, then $y_0 \in B(P(x_\beta), \delta)$ eventually so $d(y_0, P(x_\beta)) < \delta$ eventually. This contradicts (3.1). Thus $y_0 \notin A$ and we obtain (4).

To verify that (4) \Rightarrow (3), assume (4) holds and let $y_0 \in P(x_0)$. Choose $A(y_0)$ and $x_\beta \rightarrow x_0$ as in (4). Since $y_0 \notin A(y_0)$, there exists $\delta > 0$ such that $y_0 \notin \bar{B}(A(y_0), \delta)$. Choose $\beta_0 = \beta_0(\delta)$ so that $P(x_\beta) \subseteq B(A(y_0), \delta)$ for all $\beta \geq \beta_0$. Thus

$$d(y_0, P(x_\beta)) \geq d(y_0, B(A(y_0), \delta)) =: \rho > 0$$

for all $\beta \geq \beta_0$. That is, (3) holds. ■

There is an analogous characterization of when P is k -lower semicontinuous.

LEMMA 3.2. *Let k a positive integer, and $x_0 \in X$. The following statements are equivalent.*

(1) P is not k -lsc at x_0 ;

(2) There exist k compact convex sets A_1, \dots, A_k in $P(x_0)$ with $\bigcap_{i=1}^k A_i = \emptyset$ and a net $(x_{1,\beta}, \dots, x_{k,\beta})$ in X^k convergent to (x_0, \dots, x_0) such that

$$h(P(x_{i,\beta}), A_i) \rightarrow 0 \quad (i = 1, 2, \dots, k). \quad (3.3)$$

Proof. (1) \Rightarrow (2). If (1) holds, then there exist $\varepsilon > 0$ and net $(x_{1,\alpha}, \dots, x_{k,\alpha})$ in X^k convergent to (x_0, \dots, x_0) such that

$$\bigcap_{i=1}^k B(P(x_{i,\alpha}), \varepsilon) = \emptyset \quad \text{for all } \alpha.$$

Since $(P(x_{i,\alpha}))$ is a net in $\mathcal{H}(M)$ for $i = 1, 2, \dots, k$ which is eventually bounded, the Blaschke selection theorem implies that there are a subnet, $(x_{1,\beta}, \dots, x_{k,\beta})$ say, and k compact convex sets A_1, \dots, A_k in M such that $h(P(x_{i,\beta}), A_i) \rightarrow 0$ for $i = 1, 2, \dots, k$. Thus $A_i \subseteq B(P(x_{i,\beta}), \varepsilon)$ eventually ($i = 1, 2, \dots, k$) implies that

$$\bigcap_{i=1}^k A_i \subseteq \bigcap_{i=1}^k B(P(x_{i,\beta}), \varepsilon) = \emptyset.$$

It remains to show that $A_i \subseteq P(x_0)$ for each i . Since P is upper semicontinuous, for each $\delta > 0$, $P(x_{n,i}) \subseteq B(P(x_0), \delta)$ eventually ($i = 1, 2, \dots, k$). Also, since $P(x_{i,\beta}) \rightarrow A_i$, it follows that, eventually, for each i

$$A_i \subseteq B(P(x_{i,\beta}), \delta) \subseteq (P(x_0), 2\delta).$$

Since δ was arbitrary, $A_i \subseteq \overline{P(x_0)} = P(x_0)$.

(2) \Rightarrow (1). Suppose (2) holds. Then

$$\emptyset = \bigcap_{i=1}^k A_i = \bigcap_{i=1}^k \bigcap_{\varepsilon > 0} \bar{B}(A_i, \varepsilon).$$

Since the sets $\bar{B}(A_i, \varepsilon)$ are compact, there is a finite subcollection which also has an empty intersection. That is, for some $\varepsilon > 0$,

$$\bigcap_{i=1}^k B(A_i, \varepsilon) \subseteq \bigcap_{i=1}^k \bar{B}(A_i, \varepsilon) = \emptyset.$$

By (3.3), there exists a β_0 such that $P(x_{i,\beta}) \subseteq B(A_i, \varepsilon/2)$ for all $\beta \geq \beta_0$ ($i = 1, \dots, k$). Thus

$$\bigcap_{i=1}^k B(P(x_{i,\beta}), \varepsilon/2) \subseteq \bigcap_{i=1}^k B(A_i, \varepsilon) = \emptyset.$$

This proves that P is not k -lsc at x_0 . ■

The main result of this section characterizes approximate lower semicontinuity and will be used in Section 5.

THEOREM 3.3. *Let X be a topological space and M a normed linear space of finite dimension n . If $P: X \rightarrow 2^M$ is non-empty compact convex set valued and upper semicontinuous, then the following statements are equivalent.*

- (1) P is not approximate lower semicontinuous;
- (2) P is not $(n + 1)$ -lower semicontinuous;
- (3) There exists $x_0 \in X$, $n + 1$ non-empty compact convex sets A_1, \dots, A_{n+1} in $P(x_0)$ with $\bigcap_{i=1}^{n+1} A_i = \emptyset$, and a net $(x_{1,\alpha}, \dots, x_{n+1,\alpha})$ in X^{n+1} convergent to (x_0, \dots, x_0) such that

$$h(P(x_{i,\alpha}), A_i) \rightarrow 0 \quad (i = 1, \dots, n + 1).$$

Proof. The equivalence of (1) and (2) is a consequence of [10, Theorem 2.5]. The conditions (2) and (3) are equivalent by Lemma 3.2. ■

4. THE 2-LOWER SEMICONTINUITY

In this section two characterizations are given of finite-dimensional subspaces whose metric projections are not 2-lower semicontinuous. The first is in terms of the geometry of the unit ball of X , relative to the subspace M , and allows one to visualise the condition. The characterization is illustrated by an example. The second characterization is in terms of a duality relation. It is convenient to state as a proposition the negation of the definition of 2-lower semicontinuity.

PROPOSITION 4.1. *The metric projection P_M is not 2-lower semicontinuous if and only if there exist $y_0 \in X \setminus M$, $\delta' > 0$ and sequences $(y_{j,k})_{k \geq 1}$, for $j = 1, 2$, convergent to y_0 , such that*

$$h(P_M(y_{1,k}), P_M(y_{2,k})) \geq \delta'$$

for all $k = 1, 2, \dots$. One may also require that

$$d(y_{j,k}, M) = d(y_0, M) = 1.$$

The first characterization should be thought of in terms of the set

$$\ker P_M \cap S(0, 1) = \{x \in S(0, 1) \mid (x+M) \cap B(0, 1) = \emptyset\}.$$

THEOREM 4.2. *The metric projection P_M is not 2-lower semicontinuous if and only if there exist $\delta > 0$, points x_1, x_2 in X such that $\|x_1\| = \|x_2\| = 1$ and $x_2 - x_1 \in M$, and sequences $(x_{j,k})_{k \geq 1}$ for $j = 1, 2$, convergent to x_1 and x_2 respectively, such that for $j = 1, 2$ and $k = 1, 2, \dots$,*

$$x_{j,k} \in (M + x_{j,k}) \cap \bar{B}(0, 1) \subseteq S(0, 1) \quad (4.1)$$

and

$$h((M + x_{1,k}) \cap \bar{B}(0, 1), (M + x_{2,k}) \cap \bar{B}(0, 1)) \geq \delta. \quad (4.2)$$

Proof. Suppose that P_M is not 2-lsc and that δ' and $y_0, (y_{j,k})$ for $j = 1, 2$, all distance one from M , are as in Proposition 4.1. Then

$$P_M(y_{j,k}) = \bar{B}(y_{j,k}, 1) \cap M = y_{j,k} + \bar{B}(0, 1) \cap (M - y_{j,k}) \subseteq y_{j,k} + S(0, 1).$$

Choose $x_{j,k} \in \bar{B}(0, 1) \cap (M - y_{j,k})$ for $k = 1, 2, \dots$ and $j = 1, 2$. Choosing subsequences it may be supposed that $x_{j,k} \rightarrow x_j$ as $k \rightarrow \infty$ for $j = 1, 2$. Then

$y_0 + x_j \in P_M(x_0)$ for $j = 1, 2$ and $x_2 - x_1 \in M$. Also $M - y_{j,k} = M + x_{j,k}$, so Eq. 4.1 is satisfied. Furthermore,

$$\begin{aligned} &h((M + x_{1,k}) \cap \bar{B}(0, 1), (M + x_{2,k}) \cap \bar{B}(0, 1)) \\ &\geq h(P_M(y_{1,k}), P_M(y_{2,k})) - \|y_{2,k} - y_{1,k}\| \geq \delta' - \|y_{2,k} - y_{1,k}\|. \end{aligned}$$

Thus tails of the sequences $(x_{j,k})_{k \geq 1}$, $j = 1, 2$, and $\delta = \delta'/2$ satisfy the condition of the theorem.

Conversely, suppose that δ , x_1 , x_2 and $(x_{j,k})_{k \geq 1}$, for $j = 1, 2$ satisfy the conditions. Let $m_0 = x_2 - x_1$, $y_0 = \frac{1}{2}(x_1 + x_2)$, and $y_{j,k} = x_{j,k} - \frac{1}{2}(-1)^j m_0$ for $k = 1, 2, \dots$ and $j = 1, 2$. Then $y_0 = \lim_{k \rightarrow \infty} y_{j,k}$ for $j = 1, 2$. Also,

$$y_{j,k} - \bar{B}(y_{j,k}, 1) \cap M = \bar{B}(0, 1) \cap (M + x_{j,k}) \subseteq S(0, 1),$$

so that $d(y_{j,k}, M) = 1$ and, for $j = 1, 2$,

$$P_M(y_{j,k}) = y_{j,k} - \bar{B}(0, 1) \cap (M + x_{j,k}),$$

which implies that

$$\begin{aligned} &h(P_M(y_{1,k}), P_M(y_{2,k})) \\ &\geq h((M + x_{1,k}) \cap \bar{B}(0, 1), (M + x_{2,k}) \cap \bar{B}(0, 1)) - \|y_{2,k} - y_{1,k}\| \\ &\geq \delta - \|y_{2,k} - y_{1,k}\| \rightarrow \delta \quad \text{as } k \rightarrow \infty. \end{aligned}$$

It follows from Proposition 4.1 that P_M is not 2-lsc. ■

We now give a simple example of a 3-dimensional normed linear space X and a 1-dimensional subspace M of X such that P_M is not 2-lsc and so does not have a continuous selection.

EXAMPLE (The Split-Disc Space). Consider the half-disc in \mathbb{R}^3 :

$$C = \{(\alpha, 1, \gamma) \mid \alpha \geq 0, \alpha^2 + \gamma^2 \leq 1\}$$

and the convex hull $B = \text{co}(C \cup -C)$ of C with its negative. Then the Minkowski functional of B is a norm on $X = \mathbb{R}^3$ whose unit ball is B . Consider the 1-dimensional subspace $M = \mathbb{R}(0, 1, 0)$ of X .

Taking $\sigma \in \{1, -1\}$ and $x_{\sigma,\alpha} = (\sigma\alpha, \sigma, (1 - \alpha^2)^{1/2})$ for $0 < \alpha \leq 1$, we get $(x_{\sigma,\alpha} + M) \cap B = \{x_{\sigma,\alpha}\}$. Letting $\alpha \rightarrow 0$, it follows immediately from Theorem 4.2 that P_M is not 2-lsc.

This example of a space with a one-dimensional subspace for which the metric projection has no continuous selection was given in the dissertation

of Brown [5] and it has been exploited in [3]. It was also discovered independently by Kenderov in 1986 in response to a question of the second author.

The second characterization of “not 2-lsc” is in terms of both X and X^* . The following notation will be used. For $x \in X \setminus \{0\}$, let

$$J(x) := \{x^* \in X^* \mid \|x^*\| = 1, x^*(x) = \|x\|\}.$$

Thus $J(x)$ is the set of functionals in $S_{X^*}(0, 1)$ which “peak” at x , and also is the exposed face of $\bar{B}_{X^*}(0, 1)$ determined by $x \in X$. An elementary lemma is required.

LEMMA 4.3. *If P_1 and P_{-1} are compact convex subsets of a finite-dimensional space M and $h(P_1, P_{-1}) = \delta > 0$, then there exists $\phi \in M^*$ such that $\|\phi\| = 1$,*

$$\phi(p_1) - \phi(p_{-1}) \geq \delta$$

for all $p_1 \in P_1$ and $p_{-1} \in P_{-1}$, and

$$\phi(a_1 - a_{-1}) = \|a_1 - a_{-1}\|$$

whenever $a_1 \in P_1$, $a_{-1} \in P_{-1}$ and $\|a_1 - a_{-1}\| = h(P_1, P_{-1})$.

Proof. Let $C = P_1 - P_{-1}$. Then C is convex and $C \cap B_M(0, \delta) = \emptyset$. The conclusion of the lemma is satisfied by choosing a $\phi \in S_{M^*}(0, 1)$ which separates the sets C and $B_M(0, \delta)$. ■

THEOREM 4.4. *Let X be a normed linear space and M a subspace of finite dimension n . Then P_M is not 2-lsc if and only if there exist $x_0 \in X$, $b_0 \in M$, an $n-1$ dimensional subspace L of M , and for each $\sigma \in \{1, -1\}$ sequences $x_n = x_n(\sigma) \rightarrow x_0$, $y_n = y_n(\sigma) \in L + \sigma b_0$, and $\psi_n = \psi_n(\sigma) \in J(x_n - y_n) \cap L^\perp$ such that $\psi_n(\sigma b_0) > 0$ for each $n = 1, 2, \dots$*

Proof. Suppose that P_M is not 2-lsc at x_0 . Then by Proposition 4.1 there exist $\delta > 0$ and, for each $\sigma \in \{1, -1\}$, a sequence $x_n(\sigma) \rightarrow x_0$ such that

$$h(P_M(x_n(1)), P_M(x_n(-1))) \geq \delta.$$

For each σ and each n choose $a_n(\sigma) \in P_M(x_n(\sigma))$ such that

$$\|a_n(1) - a_n(-1)\| = h(P_M(x_n(1)), P_M(x_n(-1))).$$

By the lemma, for each $n = 1, 2, \dots$, there exists $\phi_n \in S_{M^*}(0, 1)$ such that

$$\begin{aligned} \inf \phi_n(P_M(x_n(1)) - \sup \phi_n(P_M(x_n(-1))) &\geq \delta, \\ \phi_n(a_n(1) - a_n(-1)) = \|a_n(1) - a_n(-1)\| &\geq \delta. \end{aligned}$$

Extracting subsequences it may be supposed that the sequences (ϕ_n) , $(a_n(1))$ and $(a_n(-1))$ are convergent to $\phi \in S_{M^*}(0, 1)$, $a(1)$ and $a(-1)$ in M (actually in $P_M(x_0)$). Then $\phi(a(1) - a(-1)) = \|a(1) - a(-1)\| \geq \delta$. Subtracting $\frac{1}{2}(a(1) + a(-1))$, it may be supposed that $\frac{1}{2}(a(1) + a(-1)) = 0$. Let $b_0 = \frac{1}{2}a(1)$ and $L = \phi^{-1}(0) \subseteq M$. Then $\phi(b_0) = \|b_0\| \geq \frac{1}{2}\delta$.

It is now easily shown that, for some n_0 , for all $n \geq n_0$, for all $y \in P_M(x_n(1))$, and all $y' \in P_M(x_n(-1))$, there follows

$$\phi(y) > \phi(b_0) \geq \delta/2, \quad \phi(y') < \phi(-b_0) \leq -\delta/2. \tag{4.3}$$

Then for each $\sigma \in \{1, -1\}$, $\phi^{-1}(\sigma \|b_0\|) = L + \sigma b_0$ and $(L + \sigma b_0) \cap P_M(x_n(\sigma)) = \emptyset$, so that $d(x_n(\sigma), L + \sigma b_0) > d(x_n(\sigma), M)$. Therefore

$$P_M(x_n(\sigma)) \subseteq B(x_n(\sigma), d(x_n(\sigma), L + \sigma b_0)). \tag{4.4}$$

For each $n \geq n_0$ and each $\sigma \in \{1, -1\}$ choose $y_n(\sigma) \in P_{L + \sigma b_0}(x_n(\sigma))$ and $\psi_n = \psi_n(\sigma) \in S_{X^*}(0, 1)$ such that

$$\inf \psi_n(B(x_n(\sigma), d(x_n(\sigma), L + \sigma b_0))) \geq \sup \psi_n(L + \sigma b_0). \tag{4.5}$$

It then follows that $\psi_n(\sigma) \in J(x_n(\sigma) - y_n(\sigma)) \cap L^\perp$. Now $\{\phi' \in M^* \mid \phi(L) = \{0\}\}$ is a subspace of M^* of dimension one, so that $\psi_n(\sigma)|_{M^*} = \lambda_n(\sigma) \phi$ for some $\lambda_n(\sigma)$. By (4.4) and (4.5), for all $y \in P_M(x_n(\sigma))$,

$$\lambda_n(\sigma) \phi(y) = \psi_n(y) \geq \psi_n(\sigma b_0) = \lambda_n(\sigma) \phi(\sigma b_0).$$

It now follows by (4.3) that $\sigma \lambda_n(\sigma) > 0$ and hence

$$\psi_n(\sigma b_0) = \lambda_n(\sigma) \sigma \phi(b_0) > 0$$

for each $\sigma \in \{1, -1\}$ and all $n \geq n_0$. This proves that the condition of the theorem is satisfied.

Now suppose that $x_0, b_0, L, x_n(\sigma), y_n(\sigma)$ and $\psi_n(\sigma)$ are as in the condition of the theorem. Choose $\phi \in M^*$ such that $\phi(L) = \{0\}$ and $\phi(b_0) = \|b_0\|$. Then $\psi_n(\sigma)|_M = \lambda_n(\sigma) \phi$ for some $\lambda_n(\sigma)$ and $\sigma \lambda_n(\sigma) > 0$. If $y \in P_M(x_n(\sigma))$, then

$$\begin{aligned} \psi_n(x_n - b_0) &= \psi_n(x_n - y_n) = \|x_n - y_n\| \\ &\geq d(x_n, M) = \|x_n - y\| \geq \psi_n(x_n - y), \end{aligned}$$

so that $\psi_n(y) \geq \psi_n(b_0)$ and $\sigma\phi(y) \geq \sigma\phi(b_0) = \sigma \|b_0\|$. Therefore if $y \in P_M(x_n(1))$ and $y' \in P_M(x_n(-1))$, then $\|\phi\| \|y - y'\| \geq \phi(y - y') \geq 2 \|b_0\|$ for all $n = 1, 2, \dots$. This proves that P_M is not 2-lsc. ■

5. SPACES WITH THE PROPERTY (CS1)

We will say that a normed linear space X has the property (CS1) if whenever M is a one-dimensional subspace of X the metric projection P_M has a continuous selection or, equivalently, is 2-lower semicontinuous. Theorem 5.1 shows that the property (CS1) is equivalent to the property that P_M is also for every finite-dimensional subspace M of X . Theorem 5.3 provides a geometric characterization of spaces with the property (CS1). The spaces of type $C_0(T)$ and $L_1(\mu)$ which have property (CS1) are determined: $C_0(T)$ does so if and only if the space T is discrete (Theorem 5.5), $L_1(\mu)$ does so if and only if it has finite dimension. Finally an example is given of a three-dimensional space X having one-dimensional subspaces with metric projections that are not lower semicontinuous, but have (unique) continuous selections. In fact X has the property (CS1).

THEOREM 5.1. *A normed linear space X has the property (CS1) if and only if the metric projection P_M is approximate lower semicontinuous for every finite-dimensional subspace M of X .*

Proof. The sufficiency of the condition is contained in the statements of Section 2.

To prove the necessity of the condition it will be shown that if M is a finite-dimensional subspace of X and P_M is not also, then there exists a one-dimensional subspace L of M such that P_L is not also. Suppose that M is such a subspace. By Theorem 3.3 there exist $x_0 \in X$ and an integer $r \geq 2$ such that for some r non-empty compact convex subsets A_1, \dots, A_r of $P_M(x_0)$ and some r sequences $(x_{j,n})_{n \geq 1}$, for $j = 1, \dots, r$, each convergent to x_0 ,

$$\bigcap_{j=1}^r A_j = \emptyset \text{ and } h(P_M(x_{j,n}), A_j) < \frac{1}{n} \quad \text{for } n = 1, 2, \dots, \quad j = 1, \dots, r.$$

We choose r to be minimal so that $\bigcap_{j=2}^r A_j \neq \emptyset$. Choose $y_1 \in A_1$ and $y_2 = \dots = y_r \in \bigcap_{j=2}^r A_j$. If we translate by $\frac{1}{2}(y_1 + y_2) \in M$, so replacing x_0 by $x_0 - \frac{1}{2}(y_1 + y_2)$, etc., then we may suppose that $y_1 + y_2 = 0$. Let $L = \mathbb{R}y_1$. It will be shown that $P'_L(x_0) = \emptyset$, so that P_L is not also at x_0 .

Choose, for each $j = 1, \dots, r$ and each $n = 1, 2, \dots$, a point $y'_{j,n} \in P_M(x_{j,n})$ such that $\|y_j - y'_{j,n}\| < 1/n$. Then

$$y_j \in P_M(x_{j,n} + y_j - y'_{j,n}) = y_j - y'_{j,n} + P_M(x_{j,n}) \subseteq B(A_j, 2/n).$$

But $y_j \in L \subseteq M$, so

$$\begin{aligned} \|x_{j,n} - y'_{j,n}\| &\geq d(x_{j,n} + y_j - y'_{j,n}, L) \\ &\geq d(x_{j,n} + y_j - y'_{j,n}, M) = \|x_{j,n} - y'_{j,n}\|, \end{aligned}$$

and therefore

$$y_j \in P_L(x_{j,n} + y_j - y'_{j,n}) = L \cap P_M(x_{j,n} + y_j - y'_{j,n}) \subseteq B(A_j, 2/n).$$

For each $j = 1, \dots, r$, $x_0 = \lim_{n \rightarrow \infty} (x_{j,n} + y_j - y'_{j,n})$ and it follows that

$$P'_L(x_0) \subseteq \bigcap_{n \geq 1} \bigcap_{j=1}^r B(A_j, 2/n) = \emptyset,$$

and the proof is complete. ■

If $z \in X \setminus \{0\}$, let $[z] = \mathbb{R}z$ denote the one-dimensional subspace of X spanned by z . For $x \in X$, define

$$p(x) := \{\gamma_0 \in \mathbb{R} \mid \|x - \gamma_0 z\| = \inf_{\gamma \in \mathbb{R}} \|x - \gamma z\|\},$$

so that $P_{[z]}(x) = p(x)z$. The lemma which follows provides a characterization of 2-lower semicontinuity of $P_{[z]}$ in terms of p .

LEMMA 5.2. *Let X be a normed linear space and let $p: X \rightarrow 2^{\mathbb{R}}$ be an upper semicontinuous mapping such that $p(x)$ is a non-empty compact interval for each $x \in X$. Given any $x_0 \in X$, the following statements are equivalent.*

- (1) p is 2-lower semicontinuous at x_0 ;
- (2) If $b > a$ and $[a, b] \subseteq p(x_0)$, then there exists $\varepsilon > 0$ such that

$$\text{either } p(x) \cap (-\infty, b] \neq \emptyset \quad \text{for all } x \in B(x_0, \varepsilon),$$

$$\text{or } p(x) \cap [a, \infty) \neq \emptyset \quad \text{for all } x \in B(x_0, \varepsilon).$$

Proof. The mapping p is not 2-lsc at x_0 if and only if there exist $\delta > 0$ and sequences $(x_{j,k})_{k \geq 1}$ convergent to x_0 , for $j = 1, 2$, such that

$$h(p(x_{1,k}), p(x_{2,k})) \geq \delta \quad \text{for all } k = 1, 2, \dots$$

We may suppose, first, that

$$\sup p(x_{1,k}) \leq \inf p(x_{2,k}),$$

and, second, by selecting a subsequence and invoking the upper semicontinuity of p , that the sequences $(\sup p(x_{1,k}))_{k \geq 1}$ and $(\inf p(x_{2,k}))_{k \geq 1}$ are convergent, and, finally, that there is an interval $[a, b]$, where $b - a > \delta/2$, such that

$$p(x_{1,k}) \subseteq (-\infty, a), \quad p(x_{2,k}) \subseteq (b, \infty) \quad \text{for all } k = 1, 2, \dots$$

The inclusions can be rewritten as

$$p(x_{1,k}) \cap [a, \infty) = \emptyset, \quad p(x_{2,k}) \cap (-\infty, b] = \emptyset \quad \text{for all } k = 1, 2, \dots$$

It follows that p is not 2-lsc at x_0 if and only if the second condition of the statement is not satisfied. ■

The final characterization of spaces X which have the property (CS1) now follows in a straightforward way from this lemma.

THEOREM 5.3. *If X is a normed linear space, then the following conditions are equivalent.*

- (1) X has the property (CS1);
- (2) For each $x_0 \in X$ and $z \in X \setminus \{0\}$, if $b > a$ and

$$[a, b] \subseteq p(x) := \{\gamma_0 \in \mathbb{R} \mid \|x - \gamma_0 z\| = \inf_{\gamma \in \mathbb{R}} \|x - \gamma z\|\},$$

then there exists $\varepsilon > 0$ such that

$$\text{either } p(x') \cap (-\infty, b] \neq \emptyset \quad \text{for all } x' \in B(x, \varepsilon),$$

$$\text{or } p(x') \cap [a, \infty) \neq \emptyset \quad \text{for all } x' \in B(x, \varepsilon).$$

Remarks. It is of interest to compare the property (2) of Theorem 5.3 with Brown's property (P). Brown [6] defined a geometric property ("property (P)") which he proved was equivalent to the metric projection onto each finite-dimensional subspace being lower semicontinuous, so that (by the Michael Selection Theorem) the metric projection onto each finite-dimensional subspace has a continuous selection. It follows that property (P) implies property (CS1). It can also be seen by a direct comparison of property (P) and the condition (2) of Theorem 5.3 that the former implies the latter. Blatter, Morris, and Wulbert [1] observed that X has

property (P) if and only if the metric projection onto each *one*-dimensional subspace of X is lower semicontinuous.

Next we turn to the question of which spaces of type $C_0(T)$ and $L_1(\mu)$ have property (CS1). Let T be a locally compact Hausdorff space. By $C_0(T)$ we mean the Banach space of all real continuous functions x on T which “vanish at infinity” (i.e., $\{t \in T \mid |x(t)| \geq \varepsilon\}$ is compact for each $\varepsilon > 0$), endowed with the uniform norm: $\|x\| = \sup\{|x(t)| \mid t \in T\}$. If T is compact, then $C_0(T)$ is the space of all real continuous functions on T , and is denoted $C(T)$. T is **discrete** if every subset of T is open.

LEMMA 5.4. *Let T be a locally compact Hausdorff space. Then T is discrete if and only if each compact subset of T is finite.*

Proof. The “only if” part is obvious.

Suppose each compact subset of T is finite and let $t \in T$. Choose a compact neighborhood U of t . Then $U = \{t, t_1, \dots, t_n\}$. For $i = 1, \dots, n$, choose disjoint neighborhoods U_i of t and V_i of t_i . Then $U \cap (\bigcap_{i=1}^n U_i) = \{t\}$ is open. Thus T is discrete. ■

For any $x \in C_0(T)$, let $Z(x) = x^{-1}(0)$ and let $\partial Z(x)$ denote the boundary of $Z(x)$. Also, for any subset T_0 of T , $\text{card } T_0$ will denote the cardinality of T_0 . A function $x \in C_0(T)$ **does not change sign** at a point $t \in T$ if there is a neighborhood U of t such that either $x \geq 0$ on U or $x \leq 0$ on U .

THEOREM 5.5. *Let T be a locally compact Hausdorff space. Then the following statements are equivalent.*

- (1) $C_0(T)$ has property (CS1);
- (2) For each $x \in C_0(T)$,
 - (a) $\text{card } \partial Z(x) \leq 1$, and
 - (b) For each $t \in \partial Z(x)$, x does not change sign at t ;
- (3) T is discrete.

Proof. (1) \Leftrightarrow (2). By Lazar *et al.* [17] when T is compact, and by Deutsch, Indumathi, and Schnatz [9, Theorem 5.1] in the general case, each one-dimensional subspace of $C_0(T)$ has a continuous metric selection if and only if (2) holds for each $x \in C_0(T) \setminus \{0\}$. (Also, (2) holds trivially if $x = 0$.)

(2) \Rightarrow (3). Suppose (2) holds and $K \subset T$ is compact. It suffices by Lemma 5.4 to show that K is finite. If K is not finite, choose a sequence (s_n) of distinct points in K . Let $s_0 \in K$ be a cluster point of (s_n) . By removing one s_n if necessary, we may assume that $s_0 \notin \{s_1, s_2, \dots\}$. We now construct a sequence (t_n) of points in K and disjoint neighborhoods U_n of t_n

inductively as follows. Fix any integer $n_1 \geq 1$ and choose disjoint neighborhoods U_1 of s_{n_1} and $U_{1,0}$ of s_0 . Having chosen points s_{n_1}, \dots, s_{n_k} with disjoint neighborhoods U_1, \dots, U_k and a neighborhood $U_{k,0}$ of s_0 disjoint from $\bigcup_{i=1}^k U_i$, select an integer $n_{k+1} > n_k$ such that $s_{n_{k+1}} \in U_{k,0}$. Then choose disjoint neighborhoods U_{k+1} of $s_{n_{k+1}}$ and $U_{k+1,0}$ of s_0 such that $U_{k+1} \cup U_{k+1,0} \subseteq U_{k,0}$. Set $t_k = s_{n_k}$ ($k = 1, 2, \dots$). This yields a sequence (t_n) in K and disjoint neighborhoods (U_n) .

Let t_0 be a cluster point of (t_n) and set $A = \{t_1, t_2, \dots\}$. Then $t_0 \in \bar{A} \setminus A$. Define x on the compact set \bar{A} by

$$x(t) = \begin{cases} 0 & \text{if } t \in \bar{A} \setminus A \\ (-1)^n 2^{-n} & \text{if } t = t_n. \end{cases}$$

It is easy to see that x is continuous on \bar{A} . By the Tietze Extension Theorem, x can be extended to a function (also denoted by x) in $C_0(T)$. Clearly, for each $t \in \bar{A} \setminus A$, we have $t \in \partial Z(x)$. By (a), it follows that $\bar{A} \setminus A = \{t_0\}$; i.e., t_0 is the *only* cluster point of (t_n) . Hence (t_n) converges to t_0 . Since $x(t_n) = (-1)^n 2^{-n}$, it follows that x changes sign at t_0 . But this contradicts (b).

(3) \Rightarrow (2). If T is discrete, $\partial Z(x)$ is empty for each $x \in C_0(T)$ so (2) holds trivially. ■

If T is discrete then $C_0(T)$ has property (P) and hence property (CS1). Thus property (P) and property (CS1) coincide in $C_0(T)$.

Also, if T is a compact Hausdorff space, then clearly T is discrete if and only if T is finite. As an immediate consequence of this observation and Theorem 5.5 we obtain the following corollary.

COROLLARY 5.6. *Let T be a compact Hausdorff space. The following statements are equivalent:*

- (1) $C(T)$ has property (CS1);
- (2) T is finite;
- (3) $C(T)$ is finite-dimensional.

Next we consider the analogue of Theorem 5.5 in spaces of integrable functions. Let (T, \mathcal{S}, μ) be a measure space and let $L_1(\mu)$ denote the space of all \mathcal{S} -measurable functions x on T with

$$\|x\| := \int_T |x| d\mu < \infty.$$

Identifying two functions which are equal a.e. (μ) , $L_1(\mu)$ becomes a Banach space.

A set $A \in \mathcal{S}$ is called an **atom** if $\mu(A) > 0$ and whenever $B \in \mathcal{S}$, $B \subseteq A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. We will not distinguish between any two sets A and B in \mathcal{S} which differ only by a set of measure zero; i.e., if $\mu[(A \cup B) \setminus (A \cap B)] = 0$, we identify A and B . A set $E \in \mathcal{S}$ is σ -finite if E is the countable union of sets having finite measure. If $x \in L_1(\mu)$, the **support** of x is the set

$$\text{supp}(x) := \{t \in T \mid x(t) \neq 0\}.$$

The following lemma collects some useful facts about atoms that we need.

LEMMA 5.7. (1) For each $x \in L_1(\mu)$, $\text{supp}(x)$ is σ -finite.

(2) There are at most a countable number of atoms in a σ -finite set, and each such atom has finite measure.

(3) A measurable function x is constant a.e. (μ) on an atom A of finite measure; this value will be denoted by $x(A)$.

(4) If A is any atom with $\mu(A) = \infty$, then $x = 0$ a.e. (μ) on A for each $x \in L_1(\mu)$.

(5) If $E \in \mathcal{S}$ has the property that $0 < \mu(E) \leq \infty$ and E contains no atoms, then for each sequence of positive numbers (ε_n) , there exists a sequence of pairwise disjoint sets (E_n) in E with $0 < \mu(E_n) < \varepsilon_n$ for each n .

The statements (1), (2), and (3) seem fairly well known. Statement (5) can be proved using the same idea as in the proof of Saks's lemma [11, p. 308]. We now verify statement (4).

If $x \in L_1(\mu)$, then

$$A \cap \text{supp}(x) = \bigcup_{n=1}^{\infty} A_n, \quad \text{where } A_n = A \cap \left\{t \mid |x(t)| \geq \frac{1}{n}\right\}.$$

If $\mu(A_n) > 0$ for some n , then since A is an atom, $\mu(A_n) = \mu(A) = \infty$. But

$$\infty = \frac{1}{n} \cdot \infty \leq \int_{A_n} |x| d\mu \leq \|x\| < \infty$$

which is absurd. Hence $\mu(A_n) = 0$ for every n so that $\mu(A \cap \text{supp}(x)) = 0$. That is, $x = 0$ a.e. (μ) on A . ■

LEMMA 5.8. *The following statements are equivalent.*

- (1) $L_1(\mu)$ is finite-dimensional;
- (2) T is a union of atoms of which at most finitely many have finite measure;
- (3) $L_1(\mu)$ is the linear span of a set of characteristic functions $\{\chi_{A_1}, \dots, \chi_{A_n}\}$, where each A_i is an atom having finite measure.

Proof. (1) \Rightarrow (2). Suppose (2) fails. Then either

- (i) T contains an infinite number of disjoint atoms with finite measure, or
- (ii) T contains a set E which has no atoms and $\mu(E) > 0$.

Using Lemma 5.7(5), it follows that in either case there exists a disjoint sequence of sets (E_n) with $0 < \mu(E_n) < \infty$. Then the sequence of functions $(\chi_{E_n} \mid n = 1, 2, \dots)$ is linearly independent in $L_1(\mu)$ so that $\dim L_1(\mu) = \infty$. That is, (1) fails.

(2) \Rightarrow (3). If (2) holds, $T = (\bigcup_{k=1}^n A_k) \cup E$, where each A_k is an atom having finite measure and E is a union of atoms having infinite measure. By Lemma 5.7(4), it follows that $x = 0$ a.e. (μ) on E for each $x \in L_1(\mu)$. Also, x is constant a.e. (μ) on each A_k implies that, for each $x \in L_1(\mu)$,

$$x = \sum_{k=1}^n x(A_k) \chi_{A_k}.$$

That is, $L_1(\mu)$ is spanned by $\{\chi_{A_1}, \dots, \chi_{A_n}\}$ and (3) holds.

The implication (3) \Rightarrow (1) is obvious. ■

A set $E \in \mathcal{S}$ is called a **unifat** [9] if it is the union of a finite number of atoms. The following intrinsic characterization of the one-dimensional subspaces in $L_1(\mu)$ which have continuous metric selections was given in [9].

THEOREM 5.9 [9, Theorem 6.3]. *Let $x \in L_1(\mu) \setminus \{0\}$. Then $P_{[x]}$ has a continuous selection if and only if whenever $A \subseteq \text{supp}(x)$ satisfies*

$$\int_A |x| d\mu = \int_{\text{supp}(x) \setminus A} |x| d\mu,$$

either A or $\text{supp}(x) \setminus A$ must be a unifat.

THEOREM 5.10. *The following statements are equivalent.*

- (1) $L_1(\mu)$ has property (CSI);
- (2) $L_1(\mu)$ is finite-dimensional.

Proof. (1) \Rightarrow (2). If $\dim L_1(\mu) = \infty$, then Lemmas 5.8 and 5.7(5) imply that there is a sequence of disjoint sets (E_n) in \mathcal{S} with $0 < \mu(E_n) < \infty$.

Define $A_n = E_{2n-1}$, $B_n = E_{2n}$, $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, and

$$x = \sum_{n=1}^{\infty} [2^n \mu(A_n)]^{-1} \chi_{A_n} + \sum_{n=1}^{\infty} [2^n \mu(B_n)]^{-1} \chi_{B_n}.$$

Then $x \in L_1(\mu) \setminus \{0\}$, A and B are disjoint, $\text{supp}(x) = A \cup B$, neither A nor B is a unifat, and

$$\int_A |x| d\mu = \sum_{n=1}^{\infty} 2^{-n} = \int_B |x| d\mu.$$

By Theorem 5.9, $P_{[x]}$ fails to have a continuous selection; that is, (1) fails.

(2) \Rightarrow (1). If $L_1(\mu)$ is finite-dimensional, then by Lemma 5.8, the support of each $x \in L_1(\mu)$ is a unifat. Thus $P_{[x]}$ has a continuous selection by Theorem 5.9. Hence $L_1(\mu)$ has property (CS1). ■

It is a consequence of Corollary 5.6 and Theorem 5.10 that if a space $C(T)$ or $L_1(\mu)$ has the property (CS1), then it is finite-dimensional and its unit ball is a polytope; it therefore has the property (P) and for every subspace M the metric projection P_M is lower semicontinuous. We conclude this section with an example of a three-dimensional space which has the property (CS1), but in which there are one-dimensional subspaces with metric projections that are not lower semicontinuous.

EXAMPLE. Let $X = l_2(2) \oplus_1 \mathbb{R}$. Thus X is \mathbb{R}^3 with a norm for which the closed unit ball B is a double cone of circular base

$$\Gamma = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 1\}$$

and vertices $(0, 0, 1)$ and $(0, 0, -1)$.

A one-dimensional subspace L of X is a Chebyshev subspace if and only if it is not parallel to a non-degenerate line segment of the unit sphere of X ; that is, if it is not parallel to a generator of the cones. There are thus two cases: (i) L is Chebyshev and P_L is singleton valued and continuous, or (ii) L is parallel to a generator $[v, x]$ where $v \in \{(0, 0, 1), (0, 0, -1)\}$ and $x \in \Gamma$. In this case

$$\{y \in X \mid y \in (y+L) \cap B \subseteq S_X(0, 1)\}$$

is the union $\Gamma \cup [v, x] \cup [-v, -x]$ of the circle Γ and two line segments each attached to the circle at an end point. Thus one sees that the metric

projection P_L is not lower semicontinuous, but that it has a unique continuous selection. Thus the space X has the property (CS1), and, furthermore, for each one-dimensional subspace of X the metric projection has a unique continuous selection.

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